*On the Asymptotic Character of Functions Defined by Series of the Form $\Sigma c_n g(x+n)$.**

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Introduction.

In a previous memoir † I have laid the foundations of a general theory of series of the forms

$$\Omega(x) = \sum_{n=0}^{\infty} c_n g(x+n), \tag{1}$$

$$\overline{\Omega}(x) = \sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)}, \tag{1'}$$

where $c_0, c_1, c_2, \ldots.$ are constants and $g(x)$ is a function of x having the asymptotic character

$$g(x) \sim h(x)\left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots.\right), \qquad h(x) = x^{P(x)} e^{Q(x)}, \tag{2}$$

$P(x)$ and $Q(x)$ being polynomials which we write in the form

$$P(x) = \mu_0 + \mu_1 x + \mu_2 x^2 + \ldots. + \mu_k x^k, \qquad \mu_k \neq 0 \text{ if } k > 0;$$
$$Q(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots. + \alpha_m x^m, \qquad \alpha_m \neq 0 \text{ if } m > 0.$$

(In case $k=0$ we assume that $m>1$.) In I we required that the asymptotic relation (2) should be valid only for x approaching infinity in a positive sense along any line whatever parallel to the axis of reals. But for the application of series $\Omega(x)$ and $\overline{\Omega}(x)$ to a study of the properties of functions in the neighborhood of a singular point, it is convenient to place a stronger condition on $g(x)$ and to require, in fact, that the asymptotic relation (2) shall be valid when x is confined in any way to a sector V including the positive axis of reals in its interior. Furthermore, we shall suppose that the singularities of $g(x)$ in the finite plane are isolated points, and that $g(x)$ is analytic in V at every point for which $|x|$ is not less than a given constant K.

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† *Transactions of the American Mathematical Society*, Vol. XVII (1916), pp. 207–232. This paper will be referred to as I. For the most part we employ here the notation of the earlier memoir.

We employ σ to denote α_m or μ_k according as m is or is not greater than k and write

$$\sigma=\sigma_0+\sigma_1\sqrt{-1},$$

where σ_0 and σ_1 are real. If $\lambda[\mu]$ is a constant determined as in Theorem XII of I, then the region of convergence [absolute convergence] of $\Omega(x)$ is bounded by the straight line $R(\sigma x)=\lambda[R(\sigma x)=\mu]$ and lies on that side of this line for which $R(\sigma x)<\lambda[R(\sigma x)<\mu]$. According to Theorem XVI of I, a given function $f(x)$ can not have more than one expansion in a series $\Omega(x)$ for a given $g(x)$, provided that $R(\sigma)$ is negative; but such an expansion is not necessarily unique when the latter condition is not satisfied. Accordingly, we shall now further restrict $g(x)$ by requiring that $R(\sigma)$ shall be negative. We shall also suppose that non-exceptional points for both $\Omega(x)$ and $\overline{\Omega}(x)$ exist in every strip parallel to the line $R(\sigma x)=0$.

The general object of this paper is to determine the asymptotic character of a function $\Omega(x)$ defined by a series of the form (1). In § 1 a somewhat extended discussion results in the fundamental Theorem I by which the asymptotic character of the function $\Omega(x)$ in relation to $g(x)$ is determined. This result is quite satisfying from the point of view of its elegance and simplicity. For the important case when $k=1$ and $m=0$ or 1, Theorem I may be stated in the somewhat more convenient form of Theorem II. The character of the convergence of the series $\overline{\Omega}(x)$ with respect to uniformity is treated in § 2, the result being stated as Theorem III.

Guided by the result contained in Theorem I, I introduce in § 3 an extension of the notion of asymptotic representation. In § 4 it is shown that a special case of this extended notion includes as a special case the notion of asymptotic representation in the sense of Poincaré. In § 5 a further treatment is given of this new asymptotic representation in the case when $k=1$, $m=0$ or 1 and $\sigma=-1$. It is shown in this case that the notion is equivalent to that of Poincaré, and explicit formulae are obtained exhibiting this equivalence. Finally, in § 6, an illustrative example is given to emphasize the fact that an $\overline{\Omega}$-series is capable of representing conveniently the asymptotic character of certain functions near infinity, while at the same time it yields also the essential properties of these functions in the finite plane.

§ 1. *Asymptotic Character of the Function $\Omega(x)$ in Relation to $g(x)$.*

At the close of I we saw incidentally that the function $\Omega(x)$ defined by the series in equation (1) has the properties expressed in the relations

$$\lim\{g(x+s)\}^{-1}\{\Omega(x)-\sum_{n=0}^{s}c_n g(x+n)\}=0, \qquad s=0,1,2,\ldots, \tag{3}$$

provided that the limits are taken for x approaching infinity in a positive sense along the axis of reals. We now make inquiry as to what less stringent restrictions may be placed on the approach of x to infinity without destroying the general validity of relations (3). Naturally we make the general restriction that x shall remain in V as it approaches infinity.

In the first place, if relations (3) are to be valid in the special case when c_1 is different from zero and all the coefficients $c_2, c_3, \ldots.$ in (1) are equal to zero, it is necessary that the ratio $g(x+1)/g(x)$ shall approach zero as x approaches infinity in the manner to be specified. Moreover, if the region over which x may vary in approaching infinity is such that $x+n$ is in the region when x is in the region, n being a positive integer, then it is easy to see that the foregoing condition is also sufficient in all cases in which the series in (1) has only a finite number of coefficients different from zero, since

$$\frac{g(x+n)}{g(x+s)}=\frac{g(x+s+1)}{g(x+s)}\cdot\frac{g(x+s+2)}{g(x+s+1)}\cdot\ldots\cdot\frac{g(x+n)}{g(x+n-1)} \tag{4}$$

when n is any integer greater than the integer s. Accordingly, we are led to ascertain first the conditions under which $g(x+1)/g(x)$ shall approach zero as x approaches infinity.

We may write

$$\frac{g(x+1)}{g(x)}\sim e^{[P(x+1)-P(x)]\log x}e^{Q(x+1)-Q(x)}e^{P(x+1)\log\left(1+\frac{1}{x}\right)}\left(1-\frac{a_1}{x^2}+\ldots.\right),$$

the relation being valid for x approaching infinity in V. From this it follows readily that $g(x+1)/g(x)$ will approach zero when and only when

$$e^{P_1(x)\log x+Q_1(x)}$$

approaches zero, x being confined to V and $P_1(x)$ and $Q_1(x)$ being polynomials with the values

$$P_1(x)=P(x+1)-P(x),$$
$$Q_1(x)=Q(x+1)-Q(x)+\frac{1}{x}p_1(x)-\frac{1}{2x^2}p_2(x)$$
$$+\frac{1}{3x^3}p_3(x)-\ldots.+(-1)^{k-1}\frac{1}{kx^k}p_k(x),$$

where $p_j(x)$ denotes the sum of the terms of degree j or greater in the function $P(x+1)$ when arranged as a polynomial in x. Moreover, if we write

$$\frac{g(x+1)}{g(x)}=e^{P_1(x)\log x+Q_1(x)}\phi(x), \tag{5}$$

then $\phi(x)$ approaches unity as x approaches infinity in V.

The leading term in the polynomial $P_1(x)$ obviously is of degree $k-1$ and has the coefficient $k\mu_k$. If $m>k$, the leading term in the polynomial $Q_1(x)$ is

of degree $m-1$ and has the coefficient $m\alpha_m$. Now, if we put $x=\rho e^{i\theta}$ where ρ is positive and write

$$R\{P_1(x)\log x+Q_1(x)\}=\pi_1(\rho)\log\rho+\pi_2(\rho),$$

we see that $\pi_1(\rho)$ and $\pi_2(\rho)$ are polynomials in ρ with coefficients involving θ. If $k\geqq m$, so that μ_k is denoted by σ, then the coefficient of the highest power of ρ in $\pi_1(\rho)$ is $k(\sigma_0\cos t\theta-\sigma_1\sin t\theta)$, where $t=k-1$. If $k<m$, so that α_m is denoted by σ, then the coefficient of the highest power of ρ in $\pi_2(\rho)$ is $m(\sigma_0\cos t\theta-\sigma_1\sin t\theta)$, where $t=m-1$. If θ is held fixed and ρ is allowed to approach $+\infty$, then the absolute value of the exponential function in (5) approaches zero or infinity according as $\sigma_0\cos t\theta-\sigma_1\sin t\theta$ is less than zero or greater than zero; and when the last quantity is equal to zero, the absolute value of the exponential quantity in (5) may approach infinity or may approach zero owing to different possible forms of $P_1(x)$ and $Q_1(x)$.

From these considerations we are led to restrict x to the largest sector V_1 which includes the positive axis of reals, is in V and is such that

$$\sigma_0\cos t\theta-\sigma_1\sin t\theta\leqq-\varepsilon \tag{6}$$

for every x in V_1, ε being a positive quantity as small as one pleases, and t being equal to $m-1$ or $k-1$ according as m is or is not greater than k. It is clear that $g(x+1)/g(x)$ approaches zero as x approaches infinity in V_1. (In case $k=1$ and $m=0$ or 1, so that $t=0$, it is obvious that ε may be chosen so that (6) is satisfied for all values of θ; in this case V_1 coincides with V.)

Returning now to a consideration of the series $\Omega(x)$, let us denote its convergence number by λ so that the boundary of its region of convergence is the straight line $R(\sigma x)=\lambda$. Let λ_1 be any number less than λ, and confine x so that $R(\sigma x)\leqq\lambda_1$. Denote by $\overline{V}$ the maximum region which is common to V_1 and the half-plane $R(\sigma x)\leqq\lambda_1$. Obviously $\overline{V}$ contains the positive axis of reals in its interior, except possibly for a segment of finite length adjacent to the point 0.

Let x_0 be a non-exceptional point for the series $\Omega(x)$ subject to the condition that

$$\lambda_1<R(\sigma x_0)<\lambda.$$

Put

$$u_n=c_n g(x_0+n),\quad v_n=\frac{g(x+n)}{g(x_0+n)},\quad n\geqq l,$$

the integer l being chosen so that $g(x_0+n)$ is different from zero for every n not less than l. Then, if we write

$$U_{l-1}=0; \quad U_n=u_l+u_{l+1}+\ldots+u_n, \quad n\geqq l,$$

we have

$$\begin{aligned}\sum_{n=l}^{m} c_n g(x+n) &= \sum_{n=l}^{m} u_n v_n \\ &= \sum_{n=l}^{m} (U_n-U_{n-1}) v_n \\ &= -\sum_{n=l}^{m} U_n(v_{n+1}-v_n)+U_m v_{m+1}.\end{aligned}$$

From the convergence of $\Omega(x_0)$ it follows that U_n approaches a finite limit as n increases indefinitely. If x is held fixed while m increases indefinitely, it is clear that V_m approaches zero. (Compare (10) of I and the discussion immediately following.) Hence it follows from the last equation that a constant M exists (independent of x) such that for every non-exceptional x in the half-plane $R(\sigma x)\leqq\lambda_1$ we have

$$\left|\sum_{n=l}^{\infty} c_n g(x+n)\right| \leqq M \sum_{n=l}^{\infty} |v_{n+1}-v_n|, \tag{7}$$

the latter series being certainly convergent as we saw in I.

In view of (1) relations (3) may be expressed in the form

$$\lim \sum_{n=s+1}^{\infty} c_n \frac{g(x+n)}{g(x+s)} = 0, \qquad s=0, 1, 2, \ldots\ldots \tag{8}$$

Let s_1 be a fixed integer and let s be any integer less than s_1. Then we may write

$$\sum_{n=s+1}^{\infty} c_n \frac{g(x+n)}{g(x+s)} = \sum_{n=s+1}^{s_1} c_n \frac{g(x+n)}{g(x+s)} + \frac{g(x+s_1)}{g(x+s)} \sum_{n=s_1+1}^{\infty} c_n \frac{g(x+n)}{g(x+s_1)}.$$

Now, suppose that x approaches infinity in $\overline{V}$ in such wise that (8) is valid for the particular value s_1 of s. Then from the last relation it follows readily that (8) is also valid, for such approach of x to infinity, whenever s is less than s_1. Therefore, to prove all the relations (8), and hence all the relations (3), valid for a given approach of x to infinity in $\overline{V}$, it is sufficient to prove them valid for those values of s which are greater than some given number.

Employing the result of the preceding paragraph and comparing (7) and (8) we see that all the relations (3) will be satisfied provided that x approaches infinity in $\overline{V}$ in such a way that

$$\lim \frac{1}{g(x+s)} \sum_{n=s+1}^{\infty} |v_{n+1}-v_n| = 0 \tag{9}$$

for all values of the integer s greater than a conveniently chosen positive integer s_1. We choose s_1 so that it is not less than l. It is obvious that (9) may be replaced by the following equivalent relations:

$$\lim \sum_{n=s+1}^{\infty} \left| \frac{g(x_0+s)}{g(x+s)} \cdot \frac{g(x+n)}{g(x_0+n)} \left\{ \frac{g(x+n+1)}{g(x_0+n+1)} \cdot \frac{g(x_0+n)}{g(x+n)} - 1 \right\} \right| = 0, \quad s>s_1. \tag{10}$$

We are now in position to prove the following theorem:

THEOREM I. *Let $\Omega(x)$ be a function defined by a series* (1) *whose convergence number λ is not $-\infty$. Let t denote $m-1$ or $k-1$ according as m is or is not greater than k. Let $\overline{V}$ denote the greatest region of the x-plane common to V, the half-plane $R(\sigma x)\leqq\lambda_1$ and that sector $\sigma_0\cos t\theta-\sigma_1\sin t\theta\leqq-\varepsilon$ which includes the positive axis of reals, where λ_1 is any number less than λ, and ε is any positive number however small and θ is defined by the relation $x=\rho e^{i\theta}$, ρ being positive. Then if x approaches infinity in $\overline{V}$ we have*

$$\lim\,\{g(x+s)\}^{-1}\{\Omega(x)-\sum_{n=0}^{s}c_n g(x+n)\}=0,\qquad s=0,1,2,\ldots\ldots$$

If in (10) we replace g in part by h it is easy to see from the foregoing results that the proof of this theorem will be complete if we show that

$$\lim\frac{h(x_0+s)}{h(x_0+s+1)}\cdot\frac{h(x+s+1)}{h(x+s)}\sum_{n=s+1}^{\infty}\left|\frac{h(x_0+s+1)h(x+n)}{h(x+s+1)h(x_0+n)}\right|\cdot\left|\frac{g(x+n+1)g(x_0+n)}{g(x+n)g(x_0+n+1)}-1\right|=0,\qquad s>s_1,\tag{11}$$

the limit being taken for x approaching infinity in $\overline{V}$; and hence if we show that the sum of the series in (11) is bounded.

For this series in (11) we use for the moment the symbol $T(x,s)$. By $\overline{T}(x,s)$ we denote the series

$$\overline{T}(x,s)=\sum_{n=s+1}^{\infty}\left|\frac{h(x_0+s+1)h(x+n)}{h(x+s+1)h(x_0+n)}\right|\cdot\left\{\left|\frac{g(x+n+1)g(x_0+n)}{g(x+n)g(x_0+n+1)}\right|+1\right\}.$$

Then the series $T(x,s)$ is term by term less than or equal to the series $\overline{T}(x,s)$.

In case k or m is greater than unity it will now be shown that $\overline{T}(x,s)$ is bounded as x approaches infinity in $\overline{V}$; and thus the proof of the theorem for this case will be completed.

Let x_1 be a non-exceptional point for the series $\Omega(x)$ such that $\lambda_1<R(\sigma x_1)<R(\sigma x_0)$. Then the series $\overline{T}(x,s)$ converges whatever the value of s (greater than s_1). Therefore, in order to complete the proof of the theorem in this case it is sufficient (compare equation (4)) to show that a number X exists such that

$$\frac{h(x+j+1)}{h(x_1+j+1)}\cdot\frac{h(x_1+j)}{h(x+j)}\tag{12}$$

is in absolute value less than 1 for every integer j, provided that $|x|>X$ and x is in $\overline{V}$. This we shall now prove without the restriction that k or m is greater than unity.

If we denote the expression in (12) by E and employ the notation $P_1(x)$ and $Q_1(x)$ introduced in the earlier part of this section, then we may write:

$$E=\frac{(x+j)^{P_1(x+j)}\left(1+\dfrac{1}{x+j}\right)^{P(x+j+1)}}{(x_1+j)^{P_1(x_1+j)}\left(1+\dfrac{1}{x_1+j}\right)^{P(x_1+j+1)}}\,e^{Q(x+j+1)-Q(x_1+j+1)-Q(x+j)+Q(x_1+j)};$$

or

$$E=(x+j)^{P_1(x+j)}(x_1+j)^{-P_1(x_1+j)}e^{Q_1(x+j)-Q_1(x_1+j)}S\equiv\overline{E}S,$$

where S is a quantity which approaches unity if either x or j approaches infinity, or if both x and j approach infinity in an independent manner, x remaining in $\overline{V}$. We may write

$$\overline{E}=\frac{|x+j|^{P_1(x+j)}\,e^{iP_1(x+j)\arg(x+j)}}{|x_1+j|^{P_1(x_1+j)}\,e^{iP_1(x_1+j)\arg(x_1+j)}}\,e^{Q_1(x+j)-Q_1(x_1+j)}.$$

We may write $P_1(x+j)$ as a polynomial in j in the form *

$$P_1(x+j)=\pi_0(x)+\pi_1(x)j+\ldots\ldots+\pi_{k-1}(x)j^{k-1}.$$

We introduce the notation

$$\overline{P}_1(x+j)=|\pi_0(x)|+|\pi_1(x)|j+\ldots\ldots+|\pi_{k-1}(x)|j^{k-1}.$$

Now, since x must remain in $\overline{V}$ it is easy to see that $|x+j|/|x_1+j|$ is bounded away from zero,† however x and j vary, provided that $|x|$ is sufficiently large. Hence a constant M exists such that

$$|\overline{E}|\leqq|x+j|^{R[P_1(x+j)]+\overline{P}_1(x_1+j)}e^{M[\overline{P}_1(x+j)+\overline{P}_1(x_1+j)]}e^{R[Q_1(x+j)-Q_1(x_1+j)]}\tag{13}$$

provided that $|x|$ is sufficiently large.

It is convenient now to separate into cases owing to the relative magnitude of k and m.

First, let us suppose that $k\geqq m$. Then (as we have seen) the leading term of $P_1(x)$, considered as a polynomial in x, is $k\sigma x^{k-1}$. With this in hand, let us consider $P_1(x+j)$ as a polynomial in j whose coefficients are arranged

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*The treatment in the text applies strictly only when $k>0$; but with the understanding that $P_1(x)\equiv 0$ and $\overline{P}_1(x)\equiv 0$ when $k=0$ the formulae employed are valid.

† This is obvious in case σ_1 is zero or j is bounded. When σ_1 is not zero and j is not bounded it may be proved as follows: Take j greater than $|x_1|$. Then the ratio in consideration is greater than $\frac{1}{4}$ if $u\geqq-\frac{1}{2}j$. When u is negative the relation $R(\sigma x)\leqq\lambda_1$ (which is satisfied when x is in $\overline{V}$) may be written in the form $\sigma_0 u-\sigma_1 v\leqq\lambda_1$ and obviously leads to the relation

$$-u\leqq\left|\frac{\sigma_1}{\sigma_0}\right||v|+\left|\frac{\lambda_1}{\sigma_0}\right|,$$

since σ_0 is negative. If $u<-\frac{1}{2}j$ we have

$$|v|>\tfrac{1}{2}\left|\frac{\sigma_0}{\sigma_1}\right|j-\left|\frac{\lambda_1}{\sigma_1}\right|,$$

whence we conclude at once to the desired result when $u<-\frac{1}{2}j$. This completes the proof.

as polynomials in ρ, where ρ is defined as in the theorem. It is clear that the highest power of ρ in the coefficient of each power of j has a coefficient whose real part is negative, since

$$\sigma_0 \cos s\theta - \sigma_1 \sin s\theta \leqq -\varepsilon$$

for each value of s in the set $s=0, 1, 2, \ldots, t$. Now, for any given positive number K_1, however large, it is clear that a constant X_1 exists such that $|x+j|>e^{K_1}$ for every j and every x such that $|x|\geqq X_1$. Hence, if we replace $|x+j|$ in (13) by e^{K_1}, having properly chosen K_1 as a quantity independent of x and j, and write the resulting relation in the form

$$|\overline{E}|<e^{\pi(\rho, j)},$$

it is clear that $\pi(\rho, j)$ is a polynomial in j whose coefficients are all negative provided that ρ is greater than an appropriately chosen constant X_2. Hence $|\overline{E}|<1$ for every j provided that $|x|\geqq X_2$. Hence the expression in (12) has the desired property of being less than 1 in absolute value for every j greater than s_1, provided only that $|x|$ is greater than an appropriately chosen constant X.

In the case in which $k<m$ the same conclusion may readily be obtained in a similar manner. In this case the polynomial $Q_1(x+j)$ plays the leading rôle and in a corresponding way, since the leading term in $Q_1(x)$, considered as a polynomial in x, is $m\sigma x^{m-1}$. In carrying out the argument it is convenient to replace $|x+j|$ by $e^{K_2(j+|u|+|v|)}$ and to observe that the latter quantity is greater than the former, however small the constant K_2 may be, provided only that $|x|$ is sufficiently large. It is easy to supply the detailed argumentation requisite here.

This completes the proof of the theorem for the case in which k or m is greater than unity.

For the case which remains we have $k=1$ and $m=0$ or 1.* Moreover, we have seen that it is sufficient to know that the series $T(x, s)$ is bounded for x approaching infinity in $\overline{V}$. But the sum of any finite number of terms of $T(x,s)$ approaches a finite value as x thus approaches infinity. Omitting $r-1$ terms of $T(x, s)$ we see that it is sufficient to our purpose to know that the quantity

$$\sum_{n=s+r}^{\infty} \left| \frac{h(x_0+s+1)\,h(x+n)}{h(x+s+1)\,h(x_0+n)} \right| \cdot \left| \frac{g(x+n+1)\,g(x_0+n)}{g(x+n)\,g(x_0+n+1)} - 1 \right| \tag{14}$$

is bounded as x approaches infinity in $\overline{V}$. The positive integer r which appears in this expression may be chosen to suit our convenience.

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* If λ_1 were chosen so as to be less than $\lambda+1/R(\sigma)$ and the definition of $\overline{V}$ were modified accordingly, the theorem could be proved in this case just as in the preceding. But for the stronger form of the theorem a modified argument is necessary.

For the case now in consideration $h(x)$ has the special value

$$h(x)=x^{\mu_0+\sigma x}e^{\alpha_0+\beta x}.$$

Then, from the relation between $h(x)$ and $g(x)$, it follows that $g(x+1)/g(x)$ may be written in the form

$$\frac{g(x+1)}{g(x)}=x^{\sigma}e^{\sigma+\beta}\left(1+\frac{\xi(x)}{x}\right), \tag{15}$$

where the function $\xi(x)$ is bounded as x approaches infinity in $\overline{V}$. By means of this relation we readily have

$$\frac{g(x+n+1)\,g(x_0+n)}{g(x+n)\,g(x_0+n+1)}-1=\frac{(x+n)^{\sigma}}{(x_0+n)^{\sigma}}-1+\frac{(x+n)^{\sigma}}{(x_0+n)^{\sigma}}\cdot\frac{r(x+n)-r(x_0+n)}{1+r(x_0+n)},$$

where $r(x)=\xi(x)/x$. From these relations and the fact that $|(x+n)^{\sigma}/(x_0+n)^{\sigma}|$ is bounded it is easy to see that the quantity in (14) is bounded provided that the same is true of each of the quantities

$$\left.\begin{aligned}&\sum_{n=s+r}^{\infty}\left|\frac{h(x_0+s+1)\,h(x+n)}{h(x+s+1)\,h(x_0+n)}\right|\cdot\left|\frac{(x+n)^{\sigma}}{(x_0+n)^{\sigma}}-1\right|,\\&\sum_{n=s+r}^{\infty}\left|\frac{h(x_0+s+1)\,h(x+n)}{h(x+s+1)\,h(x_0+n)}\right|\cdot\frac{1}{n}.\end{aligned}\right\} \tag{16}$$

That the latter of these two expressions is bounded may be proved by the method employed in the preceding case in the derivation of a similar result. Hence it remains to be shown that the first expression in (16) is bounded as x approaches infinity in $\overline{V}$.

Now the first series in (16) may be written in the form

$$\sum_{n=s+r}^{\infty}\left|\frac{h(x_0+s+r)\,h(x+n)}{h(x+s+r)\,h(x_0+n)}\right|\cdot\left|\frac{h(x_0+s+1)\,h(x+s+r)}{h(x+s+1)\,h(x_0+s+r)}\right|\cdot\left|\frac{(x+n)^{\sigma}}{(x_0+n)^{\sigma}}-1\right|.$$

Then from the form of $h(x)$ it is easy to see that the sum of this series is bounded provided that the same is true of the sum of the series

$$\sum_{n=s+r}^{\infty}\left|\frac{h(x_0+s+r)\,h(x+n)}{h(x+s+r)\,h(x_0+n)}\right|\cdot|S(x,n)|, \tag{17}$$

where $S(x,n)$ denotes the quantity

$$S(x,n)=(x-x_0)^{(r-1)\sigma}\left\{\frac{(x+n)^{\sigma}}{(x_0+n)^{\sigma}}-1\right\}=(x-x_0)^{(r-1)\sigma}\left\{\left(1+\frac{x-x_0}{x_0+n}\right)^{\sigma}-1\right\}.$$

Now, if $|x-x_0|<|x_0+n|$ and x is in $\overline{V}$ it is easy to see through the application of the binomial theorem to the last expression for $S(x,n)$ that a constant M_1 exists such that

$$|S(x,n)|<\frac{M_1}{n}.$$

Again, the quantity enclosed in braces in the first expression for $S(x, n)$ is bounded, however x and n may vary provided only that $|x|$ is sufficiently large, since under such conditions $|x+n|/|x_0+n|$ is bounded away from zero (as we saw above). Hence, if $|x-x_0|\geqq|x_0+n|$ and x is in $\overline{V}$, it follows readily that a constant M_2 exists such that

$$|S(x, n)|<\frac{M_2}{n},$$

provided that the integer r is such that $r-1\geqq-1/R(\sigma)$. We choose r in such manner as to satisfy this relation.

Combining these two inequalities for $|S(x, n)|$ we see that a constant M_3 exists such that $|S(x, n)|<M_3/n$ for every x and n provided only that n and $|x|$ are sufficiently large and x is in $\overline{V}$. Hence the sum of the series in (17) is bounded provided that the same is true of the sum of the series

$$\sum_{n=s+r}^{\infty}\left|\frac{h(x_0+s+r)h(x+n)}{h(x+s+r)h(x_0+n)}\right|\cdot\frac{1}{n}.$$

That this sum is bounded may be proved by the method employed in the preceding case in the derivation of a similar result.

This completes the proof of Theorem I.

By $\omega(x)$ and $\overline{\omega}(x)$ we denote the series

$$\omega(x)=\sum_{n=0}^{\infty}c_n g_1(x+n), \tag{18}$$

$$\overline{\omega}(x)=\sum_{n=0}^{\infty}c_n\frac{g_1(x+n)}{g_1(x)}, \tag{18$'$}$$

where $g_1(x)$ is that special case of $g(x)$ in which $k=1$ and $m=0$ or 1. For the special case $\overline{\omega}(x)$ of the series $\overline{\Omega}(x)$ the foregoing theorem may readily be put in the following interesting form:

THEOREM II. *Let $\overline{\omega}(x)$ be a function defined by a series* (18′) *whose convergence number λ is not $-\infty$. Let $\overline{V}$ denote the greatest region of the x-plane common to V and the half-plane $R(\sigma x)\leq\lambda_1$, λ_1 being a constant which is less than λ. Then, if x approaches infinity in $\overline{V}$, we have*

$$\lim x^{-s\sigma}\left\{\overline{\omega}(x)-\sum_{n=0}^{s}c_n\frac{g_1(x+n)}{g_1(x)}\right\}=0, \qquad s=0, 1, 2, \ldots\ldots \tag{19}$$

For the proof of the theorem it is sufficient to observe that

$$\frac{g_1(x+s)}{x^{s\sigma}g_1(x)}$$

approaches a finite quantity as x approaches infinity in $\overline{V}$, a result which follows readily from (15), or directly from the asymptotic form of $g_1(x)$.

For an important range of cases the region $\overline{V}$ in Theorem II may be replaced by the half-plane $R(\sigma x) \leqq \lambda_1$, namely, those cases in which the rays bounding the sector V (except for their common point zero) lie entirely within the half-plane $R(\sigma x) \geqq 0$. In these cases relations (19) are valid for x approaching infinity in any half-plane lying within the half-plane of convergence of the series $\overline{\omega}(x)$. In particular, this is the case when $g_1(x)$ is the first principal solution of the difference equation

$$f(x+1) = A(x) f(x),$$

in which $A(x)$ is a function which is analytic at infinity and has there a zero of the first order. The series $\overline{\omega}(x)$, defined by means of such a function $g_1(x)$, are of prime importance as we shall show in a later paper. For the special case when $A(x) = 1/x$ the series $\overline{\omega}(x)$ is the factorial series. Theorem II for the special case of factorial series is due to Nörlund.* The demonstration employed by Nörlund in the special case does not seem to be applicable for deriving the general theorems here obtained.

§ 2. *Uniformity of Convergence of the Series $\overline{\Omega}(x)$.*

From relation (7) it follows that constants N_1 and N_2 exist such that

$$\left.\begin{aligned} \left| \sum_{n=s+1}^{\infty} c_n \frac{g(x+n)}{g(x)} \right| &\leqq N_1 \sum_{n=s+1}^{\infty} \left| \frac{g(x+n)}{g(x)\,g(x_0+n)} \right| \cdot \left| \frac{g(x+n+1)\,g(x_0+n)}{g(x+n)\,g(x_0+n+1)} - 1 \right| \\ &\leqq N_2 \sum_{n=s+1}^{\infty} \left| \frac{h(x+n)\,h(x_0+r)}{h(x+r)\,h(x_0+n)} \right| \cdot \left| \frac{h(x+r)\,h(x_0)}{h(x)\,h(x_0+r)} \right| \\ &\qquad \cdot \left| \frac{g(x+n+1)\,g(x_0+n)}{g(x+n)\,g(x_0+n+1)} - 1 \right| \end{aligned}\right\} \quad (20)$$

for every integer s greater than 1, r being a positive integer less than s. When $k=1$ and $m=0$ or 1, one may show, by the method employed in the latter part of the proof of Theorem I in the preceding paragraph, that a constant N_3 exists such that

$$\left| \sum_{n=s+1}^{\infty} c_n \frac{g(x+n)}{g(x)} \right| \leqq N_3 \sum_{n=s+1}^{\infty} \left| \frac{h(x+n)\,h(x_0+r)}{h(x+r)\,h(x_0+n)} \right| \cdot \frac{1}{n}, \quad (21)$$

provided that $r > 1/R(\sigma)$ and s is greater than r.

Now, let x_1 be a non-exceptional point for the series $\overline{\Omega}(x)$ such that $\lambda_1 < R(\sigma x_1) < R(\sigma x_0)$. Denote by (20_1) the relation obtained from (20) by replacing the last expression in absolute value signs by the sum of the absolute values of its terms. Then the series in (20_1) dominates that in (20). But if either k or m is greater than unity the series in (20_1) converges when x is

* *Acta Mathematica*, Vol. XXXVII (1914), pp. 327–387.

replaced by x_1 and this resulting series dominates (20_1) itself, at least if $|x|$ is sufficiently large and x is in $\overline{V}$, as one sees through aid of the fact that the quantity in (12) is less than 1 in absolute value when $|x|$ is sufficiently large. If $k=1$ and $m=0$ or 1, the series in (21) converges when x is replaced by x_1, and as before, the resulting series dominates that in (21) itself, at least if $|x|$ is sufficiently large and x is in $\overline{V}$. Hence, in either case, there exists a convergent series of constant terms $\gamma_1+\gamma_2+\gamma_3+\ldots.$, such that we have the term by term inequality

$$\sum_{n=s+1}^{\infty}\left|c_n\frac{g(x+n)}{g(x)}\right|<\gamma_{s+1}+\gamma_{s+2}+\ldots., \tag{22}$$

provided that $s>1$ and $s>r+1$, at least if $|x|$ is sufficiently large and x is in $\overline{V}$.

Now, in Theorem III of *I*, we saw that $\overline{\Omega}(x)$ converges uniformly in any closed domain D which lies within its region of convergence and contains no points which are exceptional for this series, or are limit points of points which are exceptional for this series. From this result and relation (22), we conclude at once to the following theorem:

THEOREM III. *Let S be any region which lies in $\overline{V}$* and does not contain either in its interior or on its boundary a point which is exceptional for the series $\overline{\Omega}(x)$ or is a limit point of points which are exceptional for the series $\overline{\Omega}(x)$. The series $\overline{\Omega}(x)$ converges uniformly in S.*

This theorem for the special case of factorial series is due to Nörlund (*l. c.*). Nörlund's method of demonstration in the special case does not seem to be applicable for deriving the general theorem here obtained.

§ 3. *Extension of the Notion of Asymptotic Representation.*

In the complex x-plane let D be any region which extends to infinity, as for instance a sector bounded by two rays from zero to infinity. A function $f(x)$ is said to have in D the Poincaré asymptotic power series representation

$$f(x)\sim c_0+\frac{c_1}{x}+\frac{c_2}{x^2}+\ldots., \tag{23}$$

provided that

$$\lim x^s\left\{f(x)-\left(c_0+\frac{c_1}{x}+\ldots.+\frac{c_s}{x^s}\right)\right\}=0, \qquad s=0, 1, 2, \ldots.,$$

the limit in each case being taken for x approaching infinity in D. The power series in (23) may be either convergent or divergent.

* For the definition of the region $\overline{V}$ see Theorem I.

In analogy with this definition and guided by the result contained in Theorem I, it is natural to introduce an extended notion of asymptotic representation. Accordingly, we shall say that a function $f(x)$ has in D the asymptotic Ω-representation

$$f(x) \sim \sum_{n=0}^{\infty} c_n g(x+n), \tag{24}$$

with respect to $g(x)$ provided that

$$\lim \{g(x+s)\}^{-1}\{f(x) - \sum_{n=0}^{s} c_n g(x+n)\} = 0, \qquad s=0, 1, 2, \ldots,$$

the limit in each case being taken for x approaching infinity in D. Similarly we shall say that a function $f(x)$ has in D the asymptotic $\bar{\Omega}$-representation

$$f(x) \sim \sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)}, \tag{25}$$

with respect to $g(x)$ provided that

$$\lim \frac{g(x)}{g(x+s)}\left\{f(x) - \sum_{n=0}^{s} c_n \frac{g(x+n)}{g(x)}\right\} = 0, \qquad s=0, 1, 2, \ldots,$$

the limit being taken as before. For the purpose of these definitions it is not necessary that the series in (24) and (25) shall converge in D; in fact, they may be divergent for all values of x.

That the foregoing definition of asymptotic $\bar{\Omega}$-representation includes the asymptotic power series representation of Poincaré, as a special case may be seen by giving to $g(x)$ the value e^{-x^2}, and in the resulting asymptotic series (25) replacing e^{2x} by z; for the series in (25) then becomes a descending power series in z, and the factor $g(x)/g(x+s)$ is equal to the product of z^s by a quantity which is independent of z and different from zero. But this relation between the two types of asymptotic representation is relatively unimportant when compared with that which is brought to light in the next section.

It is clear that the asymptotic relations (24) and (25) are most delicate when k and m, the degrees of the polynomials $P(x)$ and $Q(x)$, are least. Consequently, it is natural to treat first the special case in which $k=1$ and $m=0$ or 1. It will indeed be seen that this case is of leading importance in the theory. Moreover, it is the case which is associated in the most valuable way with the notion of asymptotic representation from the point of view of Poincaré. Accordingly, we devote the next three sections to a treatment of this case.

§ 4. *On a Special Case of Asymptotic $\bar{\Omega}$-Representation in Relation to the Asymptotic Representation of Poincaré.*

In what follows we shall assume that the region D defined in § 3 lies in V. In the special case when $g(x)$ has the value $g_1(x)$, where

$$g_1(x) \sim x^{\mu_0+\sigma x} e^{\alpha_0+\beta x}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots\right), \tag{26}$$

the limits defining the asymptotic relation

$$f(x) \sim \sum_{n=0}^{\infty} c_n \frac{g_1(x+n)}{g_1(x)} \tag{27}$$

are equivalent to the limits

$$\lim x^{-s\sigma}\left\{f(x)-\sum_{n=0}^{s} c_n \frac{g_1(x+n)}{g_1(x)}\right\}=0, \qquad s=0,1,2,\ldots, \tag{28}$$

the limit in each case being taken for x approaching infinity in D. This follows readily from the fact that $g_1(x+s)x^{-s\sigma}/g(x)$ approaches a finite non-zero quantity as x approaches infinity in V.

We shall now obtain another set of limits equivalent to those in (28). For this purpose we observe that we have a Poincaré asymptotic representation of the form

$$\frac{g_1(x+n)}{g_1(x)} \sim x^{n\sigma}\left(\alpha_{0n}+\frac{\alpha_{1n}}{x}+\frac{\alpha_{2n}}{x^2}+\ldots\right),$$

where $\alpha_{0n}, \alpha_{1n}, \ldots$ are a set of numbers dependent only on n and the constants which enter into the asymptotic formula (26) for $g_1(x)$. Thence, we see that relations (28) are equivalent to the following:

$$\lim x^{-s\sigma}\left\{f(x)-\sum_{n=0}^{s} c_n x^{n\sigma}\left(\alpha_{0n}+\frac{\alpha_{1n}}{x}+\ldots+\frac{\alpha_{n_s n}}{x^{n_s}}\right)\right\}=0, \qquad s=0,1,2,\ldots. \tag{29}$$

Here the quantities n_s are integers dependent only on n and s. It is clear that a suitable value of n_s is any integer not less than $(s-n)(|\sigma_0|+|\sigma_1|)$. In particular, if $\sigma=-1$ we may take $n_s=s-n$; and this we do.

For the special case in which $\sigma=-1$ it is easy to see that relations (29) are equivalent to the Poincaré asymptotic formula

$$f(x) \sim \beta_0+\frac{\beta_1}{x}+\frac{\beta_2}{x^2}+\ldots, \tag{30}$$

where

$$\beta_\nu=c_0\,\alpha_{\nu 0}+c_1\,\alpha_{\nu-1,1}+c_2\,\alpha_{\nu-2,2}+\ldots+c_\nu\,\alpha_{0\nu}.$$

Hence, relation (27) with $\sigma=-1$ gives rise to the Poincaré asymptotic representation (30). In the next section we shall show that these two relations are

indeed equivalent, and shall exhibit the explicit formulae which put in evidence this equivalence. Thus we shall see that relation (27) contains (30) as a special case and shall make clear the importance of the former in view of the fundamental character of the latter.

Again, if we give to σ the rational value $-p/q$, where p and q are relatively prime positive integers, it is clear that relations (29) lead to that generalization of (30) in which x is replaced by a q-*th* root of x. In general there are necessary relations among the resulting quantities β; but if $p=1$ it is easy to see that they may have any values whatever when the quantities $c_0, c_1, c_2, \ldots$ are arbitrary.

If we give to σ an irrational real value and derive the formula corresponding to (30), we shall find that the powers of x which enter into it are not only not of integral index, but also that these indices are not integral multiples of any number whatever. This brings out the fact that (27) affords an essential generalization of (30).

On account of the leading importance of the case in which $\sigma=-1$, this case alone will be treated in the remainder of the paper.

§ 5. *Explicit Formulae Connecting Relations* (27) *and* (30) *when* $\sigma=-1$.

In the special case when $\sigma=-1$ we shall denote $g_1(x)$ by $\bar{g}(x)$, so that we have

$$\bar{g}(x) \sim x^{\mu_0-x}e^{\alpha_0+\beta x}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots\right). \tag{31}$$

We have seen that a relation of the form

$$\frac{\bar{g}(x+n)}{\bar{g}(x)} \sim x^{-n}\left(\beta_{0n}+\frac{\beta_{1n}}{x}+\frac{\beta_{2n}}{x^2}+\ldots\right) \tag{32}$$

exists, where $\beta_{0n}, \beta_{1n}, \ldots$ are a set of numbers independent of x. We shall now determine the values of these numbers β_{ij} in terms of $\mu_0, \alpha_0, \beta, a_1, a_2, \ldots$.

Setting

$$\bar{h}(x)=x^{\mu_0-x}e^{\alpha_0+\beta x},$$

we have

$$\frac{\bar{h}(x+n)}{\bar{h}(x)}=x^{-n}e^{(\beta-1)n}\left(1+\frac{\gamma_{1n}}{x}+\frac{\gamma_{2n}}{x^2}+\ldots\right), \tag{33}$$

where the constants $\gamma_{1n}, \gamma_{2n}, \ldots$ are readily determined consecutively by expanding the function in the second member of the relation

$$1+\frac{\gamma_{1n}}{x}+\frac{\gamma_{2n}}{x^2}+\ldots=\left(1+\frac{n}{x}\right)^{\mu_0-n}e^{\frac{n^2}{2x}-\frac{n^3}{3x^2}+\frac{n^4}{4x^3}-\ldots}$$

If we multiply (32) through by $\bar{g}(x)$, replace $\bar{g}(x+n)$ and $\bar{g}(x)$ by their asymptotic values from (31) and reduce the result by means of (33) we readily obtain the following relation:

$$e^{(\beta-1)n}\left(1+\frac{\gamma_{1n}}{x}+\frac{\gamma_{2n}}{x^2}+\ldots\right)\left(1+\frac{a_1}{x+n}+\frac{a_2}{(x+n)^2}+\ldots\right)$$
$$\sim\left(\beta_{0n}+\frac{\beta_{1n}}{x}+\ldots\right)\left(1+\frac{a_1}{x}+\ldots\right). \quad (34)$$

But

$$(x+n)^{-k}=x^{-k}\left(1+\frac{n}{x}\right)^{-k}$$
$$=x^{-k}\left(1-\frac{k}{1!}\cdot\frac{n}{x}+\frac{k(k+1)}{2!}\cdot\frac{n^2}{x^2}-\frac{k(k+1)(k+2)}{3!}\cdot\frac{n^3}{x^3}+\ldots\right);$$

whence we have

$$1+\frac{a_1}{x+n}+\frac{a_2}{(x+n)^2}+\ldots\sim 1+\frac{\bar{a}_1}{x}+\frac{\bar{a}_2}{x^2}+\ldots \quad (35)$$

where

$$\bar{a}_k=a_k-(k-1)na_{k-1}$$
$$+\frac{(k-1)(k-2)}{2!}n^2a_{k-2}-\frac{(k-1)(k-2)(k-3)}{3!}n^3a_{k-3}+\ldots. \quad (36)$$

Replacing the second series in (34) by its value from (35) and equating coefficients of like powers of x we have the following recurrence relations for determining $\beta_{0n}, \beta_{1n}, \ldots$:

$$\left.\begin{array}{r}\beta_{0n}=e^{(\beta-1)n},\\ \beta_{1n}+a_1\beta_{0n}=e^{(\beta-1)n}(\bar{a}_1+\gamma_{1n}),\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ \sum_{i=0}^{k}a_i\beta_{k-i,n}=e^{(\beta-1)n}\sum_{i=0}^{k}\bar{a}_{k-i}\gamma_{in},\end{array}\right\} \quad (37)$$

where $a_0=c_0=\bar{a}_0=1$. It is evident that these relations uniquely determine the quantities $\beta_{0n}, \beta_{1n}, \ldots$.

From this it follows readily that the asymptotic $\bar{\Omega}$-representation

$$f(x)\sim\sum_{n=0}^{\infty}c_n\frac{\bar{g}(x+n)}{\bar{g}(x)} \quad (38)$$

leads to the Poincaré asymptotic representation

$$f(x)\sim\beta_0+\frac{\beta_1}{x}+\frac{\beta_2}{x^2}+\ldots, \quad (39)$$

where

$$\beta_\nu=c_0\beta_{\nu 0}+c_1\beta_{\nu-1,1}+c_2\beta_{\nu-2,2}+\ldots+c_\nu\beta_{0\nu}. \quad (40)$$

On the other hand, if (39) is given then from (40) the constants $c_0, c_1, c_2, \ldots$ may be uniquely determined (since $\beta_{0\nu}\neq 0$); and hence, conversely, relation (39) leads to relation (38).

This is sufficient to establish the complete equivalence between the Poincaré asymptotic relation (39) and the asymptotic $\bar{\Omega}$-representation (38). Later we shall point out an important advantage which the latter has over the former in that the series in (38) is convergent (in a half-plane) for a large class of functions $f(x)$ for which the corresponding series in (39) is everywhere divergent.

In an important class of cases the asymptotic character of the function $\rho(x)$,

$$\rho(x)=\frac{\bar{g}(x+1)}{\bar{g}(x)},$$

is known directly and has a simple form. (It can of course be computed in the general case.) In case this asymptotic series for $\rho(x)$ is convergent, and actually represents $\rho(x)$, one has readily the descending power series expansion of $\rho'(x)/\rho(x)$. It is easy to see that it has the form

$$\frac{\rho'(x)}{\rho(x)}=-\left(\frac{1}{x}+\frac{\mu_{20}}{x^2}+\frac{\mu_{30}}{x^3}+\dots\right), \tag{41}$$

where μ_{20}, μ_{30}, ... are constants depending on the constants in $\bar{g}(x)$. In the important special case in which $\bar{g}(x)=1/\Gamma(x)$ it is easy to see that (41) takes the simple form $\rho'(x)/\rho(x)=-1/x$.

The constants β_{in} may be determined readily in terms of the constants μ_{20}, μ_{30}, ...; and the results thus obtained will sometimes be in more convenient form than those obtained by solving equations (37) for β_{in}. For effecting this determination let us write

$$y_n\equiv\frac{\bar{g}(x+n)}{\bar{g}(x)}=\rho(x)\rho(x+1)\dots\rho(x+n-1);$$

whence we have

$$\frac{y_n'}{y_n}=\frac{\rho'(x)}{\rho(x)}+\frac{\rho'(x+1)}{\rho(x+1)}+\dots+\frac{\rho'(x+n-1)}{\rho(x+n-1)}.$$

From (41) it is clear that we have an expansion of the form

$$\frac{\rho'(x+j)}{\rho(x+j)}=-\left(\frac{1}{x}+\frac{\mu_{2j}}{x^2}+\frac{\mu_{3j}}{x^3}+\dots\right).$$

Moreover, since

$$(x+j)^{-l}=x^{-l}\left(1-\frac{l}{1!}\cdot\frac{j}{x}+\frac{l(l+1)}{2!}\cdot\frac{j^2}{x^2}-\frac{l(l+1)(l+2)}{3!}\cdot\frac{j^3}{x^3}+\dots\right),$$

it is easy to see that μ_{ij} has the value

$$\mu_{ij}=\mu_{i0}-j\mu_{i-1,0}\frac{i-1}{1!}+j^2\mu_{i-2,0}\frac{(i-1)(i-2)}{2!}-\dots$$
$$+(-1)^{i-2}j^{i-2}\mu_{20}(i-1)+(-1)^{i-1}j^{i-1}. \tag{42}$$

Hence we have
$$\frac{y_n'}{y_n}=-\left(\frac{n}{x}+\frac{\varepsilon_{1n}}{x^2}+\frac{2\varepsilon_{2n}}{x^3}+\frac{3\varepsilon_{3n}}{x^4}+\ldots.\right),$$
where
$$i\varepsilon_{in}=\sum_{j=0}^{n-1}\mu_{i+1,j}. \tag{43}$$
Therefore,
$$\frac{\bar{g}(x+n)}{\bar{g}(x)}\equiv y_n=x^{-n}\beta_{0n}e^{\frac{\varepsilon_{1n}}{x}+\frac{\varepsilon_{2n}}{x^2}+\ldots..}$$
By comparison with (32) we have
$$\frac{1}{\beta_{0n}}\left(\beta_{0n}+\frac{\beta_{1n}}{x}+\frac{\beta_{2n}}{x^2}+\ldots.\right)=1+\left(\frac{\varepsilon_{1n}}{x\,!}+\frac{\varepsilon_{2n}}{x^2}+\ldots.\right)$$
$$+\frac{1}{2\,!}\left(\frac{\varepsilon_{1n}}{x}+\frac{\varepsilon_{2n}}{x^2}+\ldots.\right)^2+\ldots..$$
Thence we see that
$$\frac{\beta_{in}}{\beta_{0n}}=1+\varepsilon_{in}+\frac{1}{2\,!}\Sigma\varepsilon_{j_1n}\varepsilon_{j_2n}+\frac{1}{3\,!}\Sigma\varepsilon_{j_1n}\varepsilon_{j_2n}\varepsilon_{j_3n}+\ldots., \tag{44}$$
where in each case the summation is to be taken for varying subscripts j such that their sum in each case shall be i.

Relations (42), (43), (44) serve to express β_{in} in terms of $\mu_{20}, \mu_{30}, \ldots..$ Thence through (40) we have the relation existing between the constants in the asymptotic formulae (38) and (39).

In case $\rho(x)$ is a rational function the constants β_{in} may be determined in a very simple manner. For the sake of simplicity in the formulae we shall assume that the poles of $\rho(x)$ are of the first order, and that no two of its polar points differ in affixes by an integer. Denoting the poles of $\rho(x)$ by $p_1, p_2, \ldots., p_h$ we have a partial fraction expansion of the form
$$y_n\equiv\rho(x)\rho(x+1)\ldots.\rho(x+n-1)=\sum_{l=1}^{h}\sum_{j=0}^{n-1}\frac{\eta_{lj}}{x-p_l+j}; \tag{45}$$
whence we see readily that
$$y_n=\sum_{l=1}^{h}\sum_{j=0}^{n-1}\eta_{lj}\left(\frac{1}{x}+\frac{p_l-j}{x^2}+\frac{(p_l-j)^2}{x^3}+\ldots.\right).$$
Therefore,
$$\beta_{in}=\sum_{l=1}^{h}\sum_{j=0}^{n-1}\eta_{lj}(p_l-j)^{n+i-1}. \tag{46}$$

As an example take the case in which $\rho(x)=1/x$. Then we have
$$\frac{1}{x(x+1)\ldots.(x+n-1)}=\frac{1}{x}-\frac{1}{1\,!}\frac{1}{x+1}$$
$$+\frac{1}{2\,!}\frac{1}{x+2}-\ldots.+(-1)^{n-1}\frac{1}{(n-1)\,!}\frac{1}{x-n+1},$$
and
$$\beta_{in}=(-1)^{n+i-1}\sum_{j=0}^{n-1}(-1)^j\frac{1}{j\,!}j^{n+i-1}. \tag{47}$$

That the two asymptotic formulae (38) and (39) are equivalent in the special case in which $\rho(x)=1/x$ has been shown by Nielsen,* who obtained formulae equivalent to (47) and (40) for the expression of this equivalence.

§ 6. *Illustrative Example.*

Let us consider the function $f(x)$ defined by the series

$$f(x)=\sum_{n=0}^{\infty} c_n \frac{\bar{g}(x+n)}{\bar{g}(x)}, \tag{48}$$

where the constants c_n are subject to the condition

$$|c_n| < M(n!)^{1-\epsilon},$$

where M and ϵ are positive constants. This series is term by term less in absolute value than the series

$$\sum_{n=0}^{\infty} M(n!)^{1-\epsilon}\left|\frac{\bar{g}(x+n)}{\bar{g}(x)}\right|.$$

That the latter series converges for every non-exceptional value of x is readily shown by the Cauchy ratio test. Hence the series in (48) converges for every non-exceptional value of x. Therefore, it defines a function $f(x)$ which is analytic at every point x which is non-exceptional for the series in (48). At the exceptional points for this series $f(x)$ will in general have singularities. In particular, if $\rho(x)$ is a rational function such that its poles are of the first order and no two of its polar points differ in affix by an integer, then $f(x)$ in general will have an infinite number of poles of the first order having infinity as their sole limit point. In such a case as this the power series in (39) diverges for every value of x. It yields the asymptotic character of $f(x)$, but does not yield anything else directly. On the other hand, the series in (48) also yields the asymptotic character of $f(x)$ and—what is much more—it affords a convergent representation of $f(x)$ at all points which are non-exceptional for the series in (48); and these latter points are in general points of singularity of $f(x)$. Moreover, the series furnishes in general a ready means for investigating the character of the singularities of $f(x)$.

This example illustrates an important result which I expect to present in full in a subsequent paper. It turns out that important classes of functions exist, each function of which has an essential singularity at infinity, but nevertheless admits a convergent $\bar{\Omega}$-expansion. Moreover, as one sees from the present paper, such an $\bar{\Omega}$-expansion exhibits directly those properties of the function which in many investigations one desires first of all to derive.

University of Illinois, *April*, 1916.

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*Nielsen, *Annales Scientifiques de l'École Normale Supérieure*, Series 3, Vol. XXI (1904), pp. 449–458.